Oscillation Properties of First Order Neutral Differential Equations Near the Critical States

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1 Introduction

In the present paper we consider the first order NDE with the almost constant coefficient

$$\ell[y] := y'(t) - py'(t-1) + Q(t)y(t-\sigma) = 0, \quad t \ge t_0, \tag{1}$$

where $\sigma \ge 0$, $p = \text{const}$, and

$$\lim_{t\to\infty} Q(t) = q. \tag{2}$$

Its "limiting" equation is

$$x'(t) - px'(t-1) + qx(t-\sigma) = 0, \quad t \ge 0. \tag{3}$$

Eq.(1) some times (in the critical situations) does not inherit the oscillation properties of Eq.(3). For example, in the case $\{p = 0;\ q = \frac{1}{\sigma e}\}$ these properties depend on the character of the convergence (2) (see [5, Corollary 4.5] and [7]; compare also with the results in [2] and [3]).

The situation $\{p = 1;\ q = 0\}$ is the same. It was considered in Theorem 3.2.4 and Corollary 3.2.2 in [8] and in [1] as well. However, the case $\{p = +1;\ q = 0\}$ is special: the solutions' behavior for Eq.(1) in this situation is rather like the second order ordinary differential equations (see [1]). Other critical situations for Eq.(1) have not been studied at all.

It is to be emphasized that this is the first paper dealing with this problem in the general form.

2 The main result and some preliminary remarks

Let

$$\begin{cases} p > \dfrac{1-\sigma}{e(2-\sigma)} & \text{for } 0 \le \sigma < 2 \\ p \text{ be arbitrary} & \text{for } \sigma \ge 2 \end{cases}. \tag{4}$$

Then the equality

$$p = e^{-s}\frac{1 - s\sigma}{1 - s(\sigma - 1)} \tag{5}$$

defines the unique real number s,

$$\begin{cases} -\dfrac{1}{1-\sigma} < s < 1 & \text{for } 0 \le \sigma < 1 \\ -\infty < s < 1 & \text{for } 1 \le \sigma \le 2 \\ -\infty < s < \dfrac{1}{\sigma - 1} & \text{for } \sigma > 2 \end{cases} . \tag{6}$$

Denote

$$h[\sigma; s] := \sigma(\sigma - 1)s^2 + (1 - 2\sigma)s + 2,$$

$$\mathcal{K}[\sigma; s] := \frac{e^{-s\sigma} h[\sigma; s]}{8[1 - s(\sigma - 1)]} \text{ and } q := \frac{e^{-s\sigma} s^2}{1 - s(\sigma - 1)} . \tag{7}$$

It is easy to check that $h[\sigma; s] > 0$ in domains (6).

The following statement can be called as "Kneser-like Theorem" for the first order neutral differential equation.

Theorem 1 *Let p and $s < 1$ be defined by* (4) *and* (5)–(6). *Assume*

$$\liminf_{t \to \infty} t^2 \big[Q(t) - q\big] = C > \mathcal{K}[\sigma; s]. \tag{8}$$

Then:

1^0) *all solutions of Eq.*(1) *are oscillatory;*

2^0) *any solution of Eq.*(1) *has at least one zero on each interval $(T - 1; T \exp \frac{\pi}{\nu})$ for sufficiently large T and*

$$4\nu^2 < \frac{C}{\mathcal{K}} - 1. \tag{9}$$

Remark 1 This result is sharp in the sence that the strict inequality in (8) cannot be replaced by the non-strict one. Indeed, define

$$y_0(t) := \sqrt{t} \cdot e^{-st}, \quad p = e^{-s}\frac{1 - s\sigma}{1 - s(\sigma - 1)}, \quad Q(t) := \frac{p y_0'(t - 1) - y_0'(t)}{y_0(t - \sigma)} .$$

One can easily check that

$$\lim_{t \to \infty} t^2 \Big[Q(t) - \frac{e^{-s\sigma} \cdot s^2}{1 - s(\sigma - 1)}\Big] = \mathcal{K}[\sigma; s]. \tag{10}$$

On the other hand, Eq.(1) has a non-oscillatory solution $y_0(t)$.

Remark 2 Consider two important particular cases:

1. $\{p = 1; \sigma \ge 0\}$. Then $s = 0$, $q = 0$, $\mathcal{K}[\sigma; 0] = \frac{1}{4}$, and the first statement of Theorem 1 turns to the known result from [1] for the equation

$$x'(t) - x'(t - 1) + Q(t)x(t - \sigma) = 0.$$

The second statement of Theorem 1 is new even for this particular case;

2. $\{p = 0; \sigma > 0\}$. Then $s = \frac{1}{\sigma}$, $q = \frac{1}{e\sigma}$, $\mathcal{K}[\sigma; \frac{1}{\sigma}] = \frac{\sigma}{8e}$, and Theorem 1 turns to the known result from [7] for the equation

$$x'(t) + Q(t)x(t - \sigma) = 0.$$

3 Critical states of the autonomous first order NDE

In this section we present for the first time the complete description of the critical states of Eq.(3) with respect to its oscillation properties.

Definition 1 We say that NDE (3) is in the *critical state* with respect to its oscillation properties if there exists at least one eventually positive solution of Eq.(3), while the equation

$$z'(t) - pz'(t-1) + (q+\varepsilon)z(t-\sigma) = 0, \quad t \geq t_0$$

has oscillatory solutions only $\forall \varepsilon > 0$. The pair $\{p; q\}$ will be called a *critical pair.*

It is well-known (see, for example, [9]) that all solutions of Eq.(3) are oscillatory if and only if its characteristic equation

$$F(s) := F(\{p; q\}, s) = -s + spe^s + qe^{s\sigma} = 0 \tag{11}$$

has no real roots.

This fact gives us the possibility to discern all critical pairs for Eq.(3). Indeed, in case the pair $\{p; q\}$ is critical,

$$\exists\, \overline{s} \in (-\infty, \infty): \quad F(\overline{s}) = 0, \quad F'(\overline{s}) = 0. \tag{12}$$

Remark 3 The inverse statement is not true. The pair $\{p; q\}$ will be critical if $F(\{p; q+\varepsilon\}; s) > 0$ $\forall s \in (-\infty, \infty)$ and $\forall \varepsilon > 0$ only. Therefore, among a few pairs $\{p; q_i\}$, $i = 1, 2, \ldots, m$ given rise to the solvable system (12), the pair $[\{p; \overline{q}\} : \ \overline{q} = \max_i q_i]$ only will be critical.

Remark 4 Note that the pair $\{p; q\}$, $q < 0$ can not be critical because $F(0) = q < 0$ and $\lim_{s \to -\infty} F(s) = +\infty$. Therefore, we suppose $q \geq 0$ in (2)–(3) from the beginning.

Let the system (12) be solvable with the solution s (maybe, not unique!). Then

$$\begin{cases} F(s) = -s + spe^s + qe^{s\sigma} = 0 \\ F'(s) = -1 + p(1+s)e^s + q\sigma e^{s\sigma} = 0 \end{cases} \Longleftrightarrow \begin{cases} p = e^{-s} \dfrac{1 - s\sigma}{1 - s(\sigma - 1)} \\ q = e^{-s\sigma} \dfrac{s^2}{1 - s(\sigma - 1)} \end{cases}. \tag{13}$$

The system (13) can be considered as the parametric representation of all pairs $\{p; q\}$ so that the system (12) is solvable. In view of Remark 4 we must restrict the interval of the parameters in (13) by

$$1 - s(\sigma - 1) > 0 \Longleftrightarrow \begin{cases} -\dfrac{1}{1-\sigma} < s & \text{for } 0 \leq \sigma < 1 \\ -\infty < s < \infty & \text{for } \sigma = 1 \\ s < \dfrac{1}{\sigma - 1} & \text{for } \sigma > 1 \end{cases}. \tag{14}$$

Therefore the possible cases of all critical pairs $\{p; q\}$ of Eq.(3) look as follows:

a) Case $\sigma = 0$. There exists a critical pair (13) for $p > 0$. No critical pairs for $p \leq 0$. All points of the curve (13) present a critical pair;

b) Case $0 < \sigma < 1$. There exists a critical pair (13) for $p \geq p(s_0) = \frac{-4\sigma^2}{[1+\sqrt{1+4\sigma(1-\sigma)}]^2} e^{-s_0}$, where

$$s_0 := \frac{4}{2\sigma - 1 + \sqrt{1 + 4\sigma(1-\sigma)}}$$

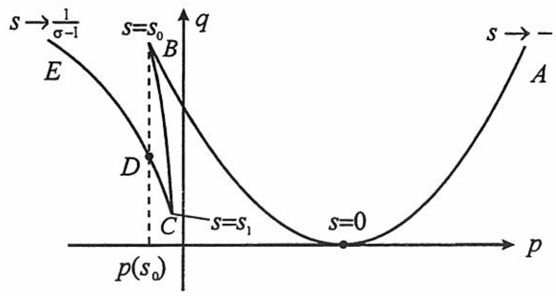
is the smallest root of the equation

$$h[\sigma; s] = 0. \tag{15}$$

No critical pairs for $p < p(s_0)$. The piece of the curve (13) coresponding to $s > s_0$ does not present any critical pair. The point $\{p(s_0), q(s_0)\}$ is an inflection point of the curve (13);

c) Case $\sigma = 1$. The analogous situation with $s_0 = 2$ and $p \geq -\frac{1}{e^2}$.

d) Case $1 < \sigma < \frac{1}{2}(1 + \sqrt{2})$. The set $(AB) \cup (CD)$ of the critical pairs $\{p; q\}$ is not connected! There exists a critical pair for any p. The number s_1 is a second root of Eq.(15). The points B and C are the inflection points of the curve (13).



e) The case $\sigma \geq \frac{1}{2}(1 + \sqrt{2})$. There exists a critical pair for any p. The set of all critical pairs is connected again.

4 Auxiliary results from Sturmian Comparison method for NDE

Further investigations are based on our approach elaborated in [4] and [6]. Below we state two basic results from [6] in an easier and convenient form (see Lemma 1 together with Theorem 2 from [6] for Theorem A and Lemma 2 together with Theorem 2 from [6] for Theorem B).

Theorem A (for the case $\sigma \neq 1$**)** *Let* $\varphi(t) > 0$ *and* $k(t)$ *be continuous functions such that for sufficiently large* t,

$$\int^{\infty} \varphi(\xi)\, d\xi = \infty, \quad \int_t^{t+\rho} \varphi(\xi)\, d\xi < \frac{\pi}{2}, \quad \rho := \max\{1; \sigma - 1\}, \tag{16}$$

$$\left.\begin{array}{ll} -\varphi(t) \operatorname{ctg} \displaystyle\int_{t-1+\sigma}^{t} \varphi(\xi)\, d\xi < k(t) < \varphi(t) \operatorname{ctg} \int_t^{t+1} \varphi(\xi)\, d\xi & \text{for } 0 \leq \sigma < 1 \\ k(t) < \varphi(t) \operatorname{ctg} \displaystyle\int_t^{t+\rho} \varphi(\xi)\, d\xi & \text{for } \sigma > 1 \end{array}\right\}. \tag{17}$$

Define $\widetilde{Q}(t)$ *by*

$$\widetilde{Q}(t + \sigma - 1) := \operatorname{cosec} \int_t^{t+\sigma-1} \varphi(\xi)\, d\xi \cdot \exp\Big(-\int_{t-1}^{t-1+\sigma} k(\xi)\, d\xi\Big) \times$$

$$\times\left[\varphi(t-1)\cos\int\limits_{t-1}^{t}\varphi(\xi)\,d\xi - k(t-1)\sin\int\limits_{t-1}^{t}\varphi(\xi)\,d\xi - p\varphi(t)\exp\int\limits_{t-1}^{t}k(\xi)\,d\xi\right] \tag{18}$$

and define $S(t)$ *by*

$$S(t-\sigma+1) := \sec\int\limits_{t}^{t+\sigma-1}\varphi(\xi)\,d\xi\cdot\exp\Big(-\int\limits_{t-1}^{t+\sigma-1}k(\xi)\,d\xi\Big)\times$$

$$\times\left[\varphi(t-1)\sin\int\limits_{t-1}^{t}\varphi(\xi)\,d\xi + k(t-1)\cos\int\limits_{t-1}^{t}\varphi(\xi)\,d\xi - pk(t)\exp\int\limits_{t-1}^{t}k(\xi)\,d\xi\right]. \tag{19}$$

Assume

$$\widetilde{Q}(t)\geq 0 \tag{20}$$

and

$$\widetilde{Q}(t)\geq S(t)\ \ \textit{for}\ \ t>T. \tag{21}$$

Then if

$$Q(t)\geq\widetilde{Q}(t),\ \ t>T, \tag{22}$$

all solutions of Eq.(1) *are oscillatory and, what is more, any solution has at least one zero on each interval* $(a-\rho, b+1)$ *for* $a>T$, *where* T *is a sufficiently large number and* $\int_a^b\varphi(\xi)\,d\xi=\pi$.

Theorem B (for the case $\sigma = 1$) *Let* $\varphi(t)>0$ *be defined as in Theorem* A, $k(t)$ *be a continuous solution of the equation*

$$L(t) := \varphi(t)\cos\int\limits_{t-1}^{t}\varphi(\xi)\,d\xi - k(t-1)\sin\int\limits_{t-1}^{t}\varphi(\xi)\,d\xi - p\varphi(t)\exp\int\limits_{t-1}^{t}k(\xi)\,d\xi = 0,\ \ t>T \tag{23}$$

such that the condition

$$k(t)<\varphi(t)\,\mathrm{ctg}\int\limits_{t}^{t+1}\varphi(\xi)\,d\xi \tag{24}$$

holds. Define $\widetilde{Q}(t)$ *by the equality*

$$\widetilde{Q}(t) := -pk(t)+\exp\Big(-\int\limits_{t-1}^{t}k(\xi)\,d\xi\Big)\left[\varphi(t-1)\sin\int\limits_{t-1}^{t}\varphi(\xi)\,d\xi + k(t-1)\cos\int\limits_{t-1}^{t}\varphi(\xi)\,d\xi\right]. \tag{25}$$

If

$$Q(t)\geq\widetilde{Q}(t)\geq 0,\ \ t>T, \tag{26}$$

then the statement of Theorem A *holds.*

5 Proof of the main result

The proof of Theorem 1 in case $\sigma \neq 1$ is based on Theorem A where we define

$$\varphi(t) := \frac{\nu}{t}, \quad k(t) := s + \frac{1}{2t} + \frac{z}{t^2}. \tag{27}$$

The number $s < 1$ is defined by (4), (5), (6), the numbers $\nu > 0$ and z will be chosen later.

Notation $f(t) \cong g(t)$ means $f(t) = g(t) + o(t^{-2})$ as $t \to \infty$.

We write down some needed asymptotics:

$$\left.\begin{array}{c}
\dfrac{t}{\nu}\varphi(t-1) \cong 1 + \dfrac{1}{t} + \dfrac{1}{t^2}; \quad \cos \displaystyle\int_{t-1}^{t} \varphi(\xi)\,d\xi \cong 1 - \dfrac{\nu^2}{2t^2}; \\
\cos \displaystyle\int_{t}^{t-1+\sigma} \varphi(\xi)\,d\xi \cong 1 - \dfrac{\nu^2(\sigma-1)^2}{2t^2}; \\
\dfrac{t}{\nu} \sin \displaystyle\int_{t-1}^{t} \varphi(\xi)\,d\xi \cong 1 + \dfrac{1}{2t} + \dfrac{2-\nu^2}{\sigma t^2}; \\
\dfrac{t}{\nu(\sigma-1)} \sin \displaystyle\int_{t}^{t-1+\sigma} \varphi(\xi)\,d\xi \cong 1 + \dfrac{\sigma-1}{2t} - \dfrac{(\sigma-1)^2(2-\nu^2)}{\sigma t^2}; \\
\cos \displaystyle\int_{t-1}^{t-1+\sigma} \varphi(\xi)\,d\xi \cong 1 - \dfrac{\nu^2\sigma^2}{2t^2}; \\
\dfrac{t}{\nu\sigma} \sin \displaystyle\int_{t-1}^{t-1+\sigma} \varphi(\xi)\,d\xi \cong 1 - \dfrac{\sigma-2}{2t} + \dfrac{2\sigma^2 - 6\sigma + 6 - \sigma^2\nu^2}{6t^2}; \\
\sec \displaystyle\int_{t}^{t+\sigma-1} \varphi(\xi)\,d\xi \cong 1 + \dfrac{\nu^2(\sigma-1)^2}{2t^2}; \\
\dfrac{\nu(\sigma-1)}{t} \operatorname{cosec} \displaystyle\int_{t}^{t+\sigma-1} \varphi(\xi)\,d\xi \cong 1 + \dfrac{\sigma-1}{2t} + \dfrac{(\sigma-1)^2(2\nu^2-1)}{12t^2}; \\
k(t-1) \cong s + \dfrac{1}{2t} + \dfrac{1+2z}{2t^2}; \\
\displaystyle\int_{t-1}^{t-1+\sigma} k(\xi)\,d\xi = s\sigma + \dfrac{\sigma}{t} + \dfrac{2\sigma - \sigma^2 + 4\sigma z}{4t^2}; \\
\exp \displaystyle\int_{t-1}^{t-1+\sigma} k(\xi)\,d\xi \cong e^{s\sigma}\Big(1 + \dfrac{\sigma}{2t} + \dfrac{4\sigma - \sigma^2 + 8\sigma z}{8t^2}\Big); \\
\exp\Big(-\displaystyle\int_{t-1}^{t-1+\sigma} k(\xi)\,d\xi\Big) = e^{-s\sigma}\Big(1 - \dfrac{\sigma}{2t} + \dfrac{-4\sigma + 3\sigma^2 - 8\sigma z}{8t^2}\Big).
\end{array}\right\} \tag{28}$$

In view of (6) it is easy to see that the conditions (16)–(17) hold for sufficiently small $\nu > 0$ and for sufficiently large t.

We omit all intermediate calculations and state the final asymptotics for $\widetilde{Q}(t)$ defined by (18) and for $S(t)$ defined by (19):

$$\widetilde{Q}(t+\sigma-1) \cong q + \frac{1}{t^2}\left\{8\mathcal{K}[\sigma;s]\cdot\frac{z}{\sigma-1} + B(s;\sigma)\cdot(1+4\nu^2)\right\}, \tag{29}$$

$$S(t+\sigma-1) \cong q + \frac{1}{t^2}\mathcal{K}[\sigma;s]\cdot(1+4\nu^2), \tag{30}$$

where q is defined in (7).

In view of $C > \mathcal{K} > 0$ one can define $\nu > 0$ and z such that $C > \mathcal{K}(1+4\nu^2)$ and

$$\mathcal{K}(1+4\nu^2) < 8\mathcal{K}\cdot\frac{z}{\sigma-1} + B(s;\sigma)\cdot(1+4\nu^2) < C. \tag{31}$$

(The exact form of the expression $B(s;\sigma)$ is not essential).

Due to (31) we obtain

$$Q(t) > \widetilde{Q}(t) > S(t) > 0 \tag{32}$$

and therefore Theorem 1 is proved based on Theorem A.

The proof of Theorem 1 in case $\sigma = 1$ is based on Theorem B and on the following

Lemma 1 *Consider on the half-axis (t_0,∞) the following non-linear integral equation*

$$u(t) = B(t)\left[1 - \exp\int_t^{t+1} u(\xi)\,d\xi\right] + A(t) := (\Phi u)(t), \tag{33}$$

where

$$|B(t)| \le C, \quad \lim_{t\to\infty} B(t) = b > 0 \tag{34}$$

and

$$\lim_{t\to\infty} t^m A(t) = 0 \ \text{ for some } m \ge 0. \tag{35}$$

Then Eq.(33) has at least one solution $u_0(t)$ such that

$$\lim_{t\to\infty} t^m u_0(t) = 0. \tag{36}$$

Proof We define the operator Φ on the Banach space $\mathbb{C} = \mathbb{C}(t_0,\infty)$ with the norm $\|u\| = \sup_{t>t_0}|u(t)|$.

Consider $\mathcal{M} := \{u \in \mathbb{C} : \ -d_1 \le u(t) \le d_2\}$, $d_1, d_2 > 0$. The bounded set $\mathcal{M}$ is convex because

$$u_1,\, u_2 \in \mathcal{M} \Longrightarrow \vartheta u_1 + (1-\vartheta)u_2 \in \mathcal{M}, \ \ 0 < \vartheta < 1.$$

Choose d_1, d_2 such that

$$\Phi\mathcal{M} \subset \mathcal{M}. \tag{37}$$

Suppose $u \in \mathcal{M}$ and $v = \Phi u$. Then

$$v(t) = A(t) + B(t)\left[1 - \exp\int_t^{t+1} u(\xi)\,d\xi\right] \le A(t) + B(t)(1 - e^{-d_1}),$$

$$v(t) \ge A(t) + B(t)(1 - e^{d_2}).$$

For $v \in \mathcal{M}$ it should be

$$\begin{cases} A(t) + B(t)[1 - e^{-d_1}] \le d_2 \\ A(t) + B(t)[1 - e^{d_2}] \ge -d_1 \end{cases} \iff$$

$$\iff A(t) + B(t)[1 - e^{-d_1}] \le d_2 \le \ln\Big[1 + \frac{A(t)}{B(t)} + \frac{d_1}{B(t)}\Big]. \qquad (38)$$

We choose d_1 and d_2 as follows: suppose $d_1 > 0$ is sufficiently large such that $b < \ln(1 + \frac{d_1}{b})$ and $b(1 - e^{-d_1}) < \ln(1 + \frac{d_1}{b})$. Then we can define d_2 by ,

$$b(1 - e^{-d_1}) < d_2 < \ln\Big(1 + \frac{d_1}{b}\Big). \qquad (39)$$

Thus, in view of (34) and (35), the inequality (38) holds for sufficiently large t_0. Therefore, $\Phi\mathcal{M} \subset \mathcal{M}$.

Now we define

$$\mathcal{M}_0 := \Big\{\forall u \in \mathcal{M} : \ \lim_{t\to\infty} t^m u(t) = 0 \ \text{uniformly}\Big\}.$$

That is $\forall \varepsilon > 0\ \exists T_\varepsilon > t_0$ such that for all $u \in \mathcal{M}_0$, $|u(t)| < \varepsilon$ for $t > T_\varepsilon$.

Evidently, $\mathcal{M}_0$ is closed in $\mathbb{C}$. Indeed, all previous reasons are valid. On the other hand, if $u \in \mathcal{M}_0$, then $\lim\limits_{t\to\infty} t^m(\Phi u)(t) = 0$,

$$\lim_{t\to\infty} t^m \Big[\exp \int_t^{t+1} u(\xi)\, d\xi - 1\Big] = 0,$$

and so, $\Phi\mathcal{M}_0 \subset \mathcal{M}_0$. Show, in addition, that the set $\{\Phi\mathcal{M}_0\}$ is compact in $\mathbb{C}(t_0, \infty)$. We have to check whether there exists a finite partition of the axis $(t_0, \infty) = \bigcup\limits_1^N I_j$ such that

$$\forall \varepsilon > 0 \ \text{and} \ \forall t_1, t_2 \in I_j : \ |v(t_1) - v(t_2)| < \varepsilon \ \text{for all} \ v \in \Phi\mathcal{M}_0.$$

According to the definition of $\mathcal{M}_0$, let T_ε be such that $|v(t)| < \frac{\varepsilon}{8C}$ for $t > T_\varepsilon$ and for any $v \in \Phi\mathcal{M}_0$. Suppose $I_j := [t_0 + (j-1)\delta, t_0 + j\delta]$, $j = \overline{1, N-1}$, $\delta := \frac{T_\varepsilon - t_0}{N-1}$, $d = \max\{d_1, d_2\}$, and $I_N := (T_\varepsilon, \infty)$.

In view of the equality $e^{z_1} - e^{z_2} = e^{\theta z_1 + (1-\theta) z_2}(z_1 - z_2)$, $0 < \theta < 1$ and the notation $z(t) := \int\limits_t^{t+1} u(\xi)\, d\xi$, we have

$$|e^{z(t_1)} - e^{z(t_2)}| \le e^{2d}|z(t_1) - z(t_2)| = e^{2d}\Big|\int_{t_1}^{t_1+1} u(\xi)\, d\xi - \int_{t_2}^{t_2+1} u(\xi)\, d\xi\Big| =$$

$$= e^{2d}\Big|\int_{t_1}^{t_2} u(\xi)\, d\xi - \int_{t_1+1}^{t_2+1} u(\xi)\, d\xi\Big| \le e^{2d}\Big|\int_{t_1}^{t_2} |u(\xi)|\, d\xi + \int_{t_1+1}^{t_2+1} |u(\xi)|\, d\xi\Big| \le$$

$$\le 2de^{2d}C|t_1 - t_2| < 2Cde^{2d}\delta$$

$\forall t_1, t_2 \in I_j$, $j = \overline{1, N-1}$.

Suppose N is sufficiently large, such that $2dCe^{2d}\delta < \frac{\varepsilon}{3}$, and $\forall t_1,\, t_2 \in (t_0, T_\varepsilon)$, $|t_1 - t_2| < \delta$, the following inequalities hold:

$$|A(t_1) - A(t_2)| < \frac{\varepsilon}{3}, \quad |B(t_1) - B(t_2)| < \frac{\varepsilon}{3(1+e^d)}.$$

Then $\forall t_1,\, t_2 \in I_j$, $j = \overline{1, N-1}$, we obtain

$$|v(t_1) - v(t_2)| \le |A(t_1) - A(t_2)| + |B(t_1) - B(t_2)| + \Bigg| B(t_1) \exp \int\limits_{t_1}^{t_1+1} u(\xi)\, d\xi -$$

$$-B(t_2) \exp \int\limits_{t_2}^{t_2+1} u(\xi)\, d\xi \Bigg| \le \frac{\varepsilon}{3} + |B(t_1) - B(t_2)| + |B(t_2) - B(t_1)| \exp \int\limits_{t_1}^{t_1+1} u(\xi)\, d\xi +$$

$$+|B(t_2)| \Bigg| \exp \int\limits_{t_2}^{t_2+1} u(\xi)\, d\xi - \exp \int\limits_{t_1}^{t_1+1} u(\xi)\, d\xi \Bigg| \le \frac{\varepsilon}{3} + \frac{\varepsilon}{3(1+e^d)} \cdot (1+e^d) + C2de^{2d}\delta < \varepsilon.$$

For $t_1,\, t_2 \in I_N$ $(t_1,\, t_2 > T_\varepsilon)$ we obtain

$$|v(t_1) - v(t_2)| \le 4C \Bigg[\int\limits_{t_1}^{t_1+1} |u(\xi)|\, d\xi + \int\limits_{t_2}^{t_2+1} |u(\xi)|\, d\xi \Bigg] < 8C \frac{\varepsilon}{8C} = \varepsilon.$$

And so, the set $\Phi \mathcal{M}_0 \subset \mathbb{C}_0$ is compact.

Thus the continuous operator Φ transforms the convex bounded set $\mathcal{M}_0 \subset \mathbb{C}$ into its compact subset. Therefore, according to Schauder's Fixed Point Theorem, Eq.(33) has a solution $u_0(t)$ such that (36) holds. Lemma 1 is proved. □

Proof of Theorem 1 for case $\sigma = 1$. We consider now the NDE

$$y'(t) - py'(t-1) + Q(t)y(t-1) = 0, \quad t \ge t_0, \tag{40}$$

where $p > 0$ and

$$\lim_{t\to\infty} Q(t) = q. \tag{41}$$

Let $s < 1$ be the (unique) root of the equation $e^{-s}(1-s) = p$ and the pair $\{p; q\}$ be critical (that is $q = s^2 \cdot e^{-s}$). Define $\varphi(t) := \frac{\nu}{t}$ in Theorem B and consider Eq.(23), in which we introduce the new variable $u(t)$ by

$$k(t) := s + \frac{1}{2t} + \frac{z}{t^2} + u(t) := k_0(t) + u(t) \tag{42}$$

and z will be chosen later.

Using the suitable asymptotics from (28) we find (see (23))

$$L_0(t) := L[k_0(t)] = -\frac{\nu}{t^2}\left[(2-s)z + \frac{(3-s)(1+4\nu^2)}{24}\right] + o(t^{-3}).$$

Choose $z := -\frac{(3-s)(1+4\nu^2)}{24(2-s)}$. Then we get $L_0(t) = o(t^{-3})$. Substituting (42) into Eq.(23) we obtain the equation relative to the variable $u(t)$:

$$\sin \int_{t-1}^{t} \varphi(\xi)\, d\xi \cdot u(t-1) = L_0(t) - p\varphi(t) \exp \int_{t-1}^{t} k_0(\xi)\, d\xi \left[\exp \int_{t-1}^{t} u(\xi)\, d\xi - 1\right].$$

The equation obtained is Eq.(33) with

$$A(t) := \operatorname{cosec} \int_{t}^{t+1} \varphi(\xi)\, d\xi \cdot L_0(t+1) = o(t^{-2}),$$

$$B(t) := p\varphi(t+1) \cdot \operatorname{cosec} \int_{t}^{t+1} \varphi(\xi)\, d\xi \exp \int_{t}^{t+1} k_0(\xi)\, d\xi, \quad \lim_{t\to\infty} B(t) = 1 - s > 0.$$

Then, according to Lemma 1, Eq.(23) has a (exact) solution of the form (42), where $u(t) = o(t^{-2})$ and, in view of $s < 1$, (24) holds. Substituting (42) into (25) and omitting all intermediate calculation, we find

$$\begin{aligned}\widetilde{Q}(t) &= e^{-s} \cdot s^2 + \frac{e^{-s}(2-s) \cdot (1+4\nu^2)}{8t^2} + o(t^{-2}) = \\ &= q + \mathcal{K}[1; s] \cdot (1+4\nu^2) \cdot \frac{1}{t^2} + o(t^{-2}).\end{aligned}$$

So, it is possible to choose $\nu > 0$ such that $C > \mathcal{K}[1; s] \cdot (1+4\nu^2)$, and therefore Theorem 1 for the case $\sigma = 1$ is proved.

References

[1] S. S. Cheng and J. U. Zhang, On a functional differential equation related to the Emden–Fowler equation. *Funct. Differ. Eqs.* **4**(1997), No. 1–2, 215–221.

[2] J. Diblik, Positive and oscillating solutions of differential equations with delay in critical case. *J. Comput. Appl. Math.* **88**(1998), 185–202.

[3] J. Diblik and N. Koksch, Positive solutions of the equation $x'(t) = -c(t)x(t-\tau)$ in the critical case. *J. Math. Anal. Appl.* **250**(2000), 635–659.

[4] Y. Domshlak, On oscillation and location of the solutions' zeros for neutral differential equations. (Russian) *Izv. Akad. Nauk Azerb.* **2**(1984), 24–30.

[5] Y. Domshlak, On oscillation properties of delay differential equations with oscillating coefficients. *Funct. Differ. Eqs.* **2**(1994), 59–68.

[6] Y. Domshlak, G. Kvinikadze, and I. P. Stavroulakis, Sturmian comparison method: the version for the first order neutral differential equations. *Math. Inequal. Appl.* **5**(2002), 247–256.

[7] Y. Domshlak and I. P. Stavroulakis, Oscillation of first order delay differential equations in a critical state. *Appl. Anal.* **61**(1996), 359–371.

[8] L. Erbe, Q. Kong, and B. G. Zhang, Oscillation theory for functional differential equations. *Marcel Dekker*, 1994.

[9] Y. G. Sficas and I. P. Stavroulakis, Necessary and sufficient conditions for oscillations of neutral differential equations. *J. Math. Anal. Appl.* **123**(1987), No. 2, 494–507.